Time-fractional diffusion equation with time dependent diffusion coefficient

Kwok Sau Fa and E. K. Lenzi
Departamento de Física, Universidade Estadual de Maringá, Avenue Colombo 5790, 87020-900, Maringá-PR, Brazil
(Received 18 April 2005; published 18 July 2005)

We consider the time-fractional diffusion equation with time dependent diffusion coefficient given by ${}_0O^{\alpha}_{(C)t}W(x,t)=D_{\alpha,\gamma}t^{\gamma}[\partial^2W(x,t)/\partial x^2]$, where ${}_0O^{\alpha}_{(C)t}$ is the Caputo operator. We investigate its solutions in the infinite and the finite domains. The mean squared displacement and the mean first passage time are also considered. In particular, for $\alpha=0$, the mean squared displacement is given by $\langle x^2\rangle \sim t^{\gamma}$ and we verify that the mean first passage time is finite for superdiffusive regimes.

DOI: 10.1103/PhysRevE.72.011107 PACS number(s): 05.40.−a, 05.60.−k, 66.10.Cb

I. INTRODUCTION

Fractional derivatives have been a common tool to investigate different branches of science. They have been employed to describe, for instance, mechanical systems, diffusion processes in physics, chemistry and biology, in financial markets and in turbulent systems [1]. In diffusion processes, the fractional derivatives have been used to construct different kinds of the fractional diffusion equations. These equations can describe subdiffusive and superdiffusive processes. For instance, they can comply with the asymptotic shape of the random walk model and Lévy flights.

Recently, the fractional diffusion equations have been intensively investigated under several aspects [1,2]. Their analytical solutions for the probability distribution, in the infinite and the finite domains, have been obtained by several techniques [2,3]. Physical quantities such as mean squared displacement and first passage time have also been investigated and applied to the physical systems. However, these analyses have been restricted to a constant diffusion coefficient.

In this work, we intend to extend these analyses. In particular, we employ the time-fractional diffusion equation with a time dependent diffusion coefficient given by $D(t) \sim t^{\gamma}$. For this case, the solutions for the probability distribution, in the infinite and the finite domains, can be obtained by employing the series expansion technique. In the infinite domain, we obtain the mean squared displacement and we also show the behaviors of some specific solutions. Whereas, in the finite domain, the probability distribution and the mean first passage time, for some specific solutions, are analyzed.

II. TIME-FRACTIONAL EQUATION IN THE INFINITE DOMAIN

Usually, the time-fractional diffusion equation has been investigated by considering a constant diffusion coefficient given by

$$\frac{\partial W(x,t)}{\partial t} = {}_0O^{1-\alpha}_{(RL)t} D_{\alpha}\frac{\partial^2 W(x,t)}{\partial x^2}, \tag{1}$$

where ${}_0O^{1-\alpha}_{(RL)t}$ is the Riemann-Liouville operator defined by

$${}_0O^{1-\alpha}_{(RL)t}W(x,t) = \frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t dt'\,\frac{W(x,t')}{(t-t')^{1-\alpha}}. \tag{2}$$

Note that Eq. (1) can be rewritten by using the Caputo operator ${}_0O^{\alpha}_{(C)t}$

$${}_0O^{\alpha}_{(C)t}W(x,t) = D_{\alpha}\frac{\partial^2 W(x,t)}{\partial x^2}, \tag{3}$$

where

$${}_0O^{\alpha}_{(C)t}W(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t dt'\,\frac{\partial W(x,t')/\partial t'}{(t-t')^{\alpha}}. \tag{4}$$

The solution of Eq. (1) or Eq. (3) can be written in terms of the Fox function or

$$W(x,t) = \frac{1}{\sqrt{4D_{\alpha}t^{\alpha}}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n\,!\,\Gamma(1-\alpha(n+1)/2)}\left(\frac{x^2}{D_{\alpha}t^{\alpha}}\right)^{n/2}. \tag{5}$$

As mentioned above, this solution has been obtained by using several methods. Basically, these methods use the transform techniques. In the case of the time dependent diffusion coefficient we employ the series expansion method, and for appropriate parameter values one can recover the solution (5) and the solution of the usual diffusion equation ($\alpha=1$). For time dependent diffusion coefficient $D(t)=D_{\alpha,\gamma}t^{\gamma}$, the time-fractional diffusion equation is given by

$$\frac{\partial W(x,t)}{\partial t} = {}_0O^{1-\alpha}_{(RL)t}\left[D_{\alpha,\gamma}t^{\gamma}\frac{\partial^2 W(x,t)}{\partial x^2}\right]. \tag{6}$$

We note that the use of the time dependent diffusion coefficient is not new and it has been employed in the usual diffusion equation to study, for instance, turbulent systems [4]. As a matter of fact there is no fundamental reason to employ solely the usual diffusion equation. As noted in [5], other approaches can also be used to investigate the physical systems. This consideration offers another perspective of coupling the spatial and temporal variables. Equation (6) can also be written in terms of the Caputo operator on the left side, i.e.,

$$ {}_0O^{\alpha}_{(C)t}W(x,t) = D_{\alpha,\gamma}t^{\gamma}\frac{\partial^2 W(x,t)}{\partial x^2}. \tag{7} $$

We see that Eqs. (6) and (7) permit us two interpretations: Eq. (7) has the similar structure of the usual diffusion equation and we may interpret $D_{\alpha,\gamma}t^{\gamma}$ as a diffusion coefficient, whereas Eq. (6) may be interpreted as a generalization of the fractional operator given by $[1/\Gamma(\alpha)](\partial/\partial t)\int_0^t dt'[t'^{\gamma}W(x,t')/(t-t')^{1-\alpha}]$.

A physical quantity that can be easily obtained from Eq. (6) or Eq. (7) is the mean squared displacement which corresponds to

$$ \langle x^2\rangle = \frac{2D_{\alpha,\gamma}\Gamma(1+\gamma)}{\Gamma(1+\alpha+\gamma)}t^{\alpha+\gamma}. \tag{8} $$

As we can see that the mean squared displacement presents the power law behavior. Subdiffusive processes arise from $0<\alpha+\gamma<1$, while superdiffusive processes arise from $\alpha+\gamma>1$. Localized processes arise from $\alpha+\gamma<0$. Note that the normal diffusive processes arise from $\alpha+\gamma=1$ which is absent in the case of the fractional diffusion equation given by Eq. (1), except for $\alpha=1$. Moreover, a complete stationary process can be formed for any time when $\gamma=-\alpha$.

The solutions of Eq. (6) or Eq. (7) can be obtained by using the series expansion given by

$$ W(x,t) = \sum_{n=0}^{\infty} A_n(-D_{\alpha,\gamma})^n t^{\beta n+a}x^n. \tag{9} $$

We note that Eq. (7) is symmetrical with respect to the x coordinate. Therefore, we can first obtain the solution for the interval $0\leqslant x<\infty$, then it is matched with that of the interval $-\infty<x\leqslant 0$. Substituting Eq. (9) into Eq. (7) we obtain the following recurrence relation:

$$ A_{n+2} = \frac{\Gamma[1+a-(\alpha+\gamma)n/2]A_n}{D^3_{\alpha,\gamma}(n+2)(n+1)\Gamma[1+a+\gamma-(\alpha+\gamma)(n+2)/2]}. \tag{10} $$

Unfortunately, the general solution can only be written in terms of the products of Gamma function which is difficult to be manipulated. However, we can verify the solution of the usual Fokker-Planck equation and the solution (5). In order to obtain the solution (5) we set $\gamma=0$. Then, we take $a=-\alpha/2$, $A_0=1/[2\sqrt{D_\alpha}\Gamma(1-\alpha/2)]$, and $A_1=1/[2D^2_\alpha\Gamma(1-\alpha)]$. The solution of the usual diffusion equation with time dependent diffusion coefficient $D_{1,\gamma}t^{\gamma}$ can be obtained from the relation (10) by taking $\alpha=1$, $a=-(1+\gamma)/2$ and $A_1=0$ which is given by

$$ W_{1,\gamma}(x,t) = \frac{A_0}{t^{(1+\gamma)/2}}\exp\left[-\frac{(1+\gamma)x^2}{4D_{1,\gamma}t^{1+\gamma}}\right]. \tag{11} $$

This last solution is in accord to the solution obtained in Ref. [6].

For $\gamma=-\alpha$ we obtain a stationary solution and it has the exponential form

$$ W_{\alpha,-\alpha}(x,t) = \sqrt{\frac{D_{\alpha,-\alpha}\Gamma[1-\alpha(3-\alpha)/2]}{4\Gamma[1-\alpha(1-\alpha)/2]}} \times\exp\left[-\sqrt{\frac{\Gamma[1-\alpha(1-\alpha)/2]}{D_{\alpha,-\alpha}\Gamma[1-\alpha(3-\alpha)/2]}}|x|\right]. \tag{12} $$

This result is in accord to the second moment (8) which is independent of the time.

For $\alpha=0$, we can also obtain the exponential solution by taking $A_1=A_0/D^{3/2}_{0,\gamma}$, then

$$ W_{0,\gamma}(x,t) = \frac{1}{\sqrt{4D_{0,\gamma}t^{\gamma}}}\exp\left[-\frac{|x|}{\sqrt{D_{0,\gamma}}t^{\gamma/2}}\right]. \tag{13} $$

Let us now compare with the solution obtained from the usual diffusion equation with variable diffusion coefficient $|x|^{-\theta}$ (see Ref. [6], and references therein) which is given by

$$ W_{FP}(x,t) = \frac{1}{2\Gamma\left[\frac{3+\theta}{2+\theta}\right][D_\theta(2+\theta)^2t]^{1/(2+\theta)}} \times\exp\left[-\frac{|x|^{2+\theta}}{D_\theta(2+\theta)^2t}\right]. \tag{14} $$

We see that the solution (13) for $\gamma=2$ has the same form of the solution (14) for $\theta=-1$. This result shows that two different approaches can access the same specific solution.

III. TIME-FRACTIONAL EQUATION IN THE FINITE DOMAIN

In diffusion processes, they are often of interest to know how long a particle remains in a certain region of space. A physical quantity which may be related to this time is the mean first passage time (MFPT). The MFPT has been analyzed in many systems where the Brownian motion is present. Recently, the study of the MFPT has been extended to other systems that exhibit anomalous regimes. We should note that the MFPT related to the time-fractional diffusion equation, with constant diffusion coefficient, diverges due to the long tail of the temporal probability distribution. So this complete immobilization time is independent of the length of the spatial interval considered. In this work, the MFPT is investigated by using the time-fractional diffusion equation (7). It is of our interest to know how the exponent γ can modify the structure of the temporal probability distribution, too. Further, application of the MFPT in one-dimensional anomalous heat conduction systems is considered. To do so, we consider a particle diffusing in a finite interval $[-a,b]$ subject to absorbing boundaries, $W(-a,t)=W(b,t)=0$, and the initial condition given by $W(x,0)=\delta(x-x_0)$. For this case, Eq. (7) can be solved by using the method of separation of variables. Let $W(x,t)=X(x)P(t)$. From Eq. (7) yields

$$ \frac{d^2X(x)}{dx^2} = -\lambda^2X(x) \tag{15} $$

and

$$ {}_0O^{\alpha}_{(C)t}P(t) = -\lambda^2 D_{\alpha,\gamma}t^{\gamma}P(t). \quad (16)$$

The solution of Eq. (15) subject to the given boundary conditions is given by

$$X(x) = A_n \sin[\lambda_n(x+a)] \quad (17)$$

with

$$\lambda_n = \frac{n\pi}{a+b}, \quad n=1,2,\ldots . \quad (18)$$

Equation (16) can be solved by using the power series method as

$$P(t) = \sum_{m=0}^{\infty} B_m(-\lambda_n^2 D_{\alpha,\gamma}t^{\beta})^m. \quad (19)$$

Substituting into Eq. (16) we obtain

$$P(t) = B_0\left[1 + \sum_{m=1}^{\infty}\left(\prod_{k=1}^{m}\frac{\Gamma[1-\alpha+k(\alpha+\gamma)]}{\Gamma[1+k(\alpha+\gamma)]}\right) \times(-\lambda_n^2 D_{\alpha,\gamma}t^{\alpha+\gamma})^m\right]. \quad (20)$$

The solution for $W(x,t)$ is now given by

$$W_{\alpha,\gamma}(x,t) = B_0\sum_{n=1}^{\infty} A_n \sin[\lambda_n(x+a)] \times\left[1 + \sum_{m=1}^{\infty}\left(\prod_{k=1}^{m}\frac{\Gamma[1-\alpha+k(\alpha+\gamma)]}{\Gamma[1+k(\alpha+\gamma)]}\right) \times(-\lambda_n^2 D_{\alpha,\gamma}t^{\alpha+\gamma})^m\right]. \quad (21)$$

The coefficients B_0A_n can be determined by imposing the initial condition $W(x,0)=\delta(x-x_0)$, we obtain

$$B_0A_n = \frac{2}{a+b}\sin\left[\frac{n\pi(x_0+a)}{a+b}\right]. \quad (22)$$

Substituting into Eq. (21), we obtain

$$W_{\alpha,\gamma}(x,t) = \sum_{n=1}^{\infty}\frac{2\sin[\lambda_n(x_0+a)]\sin[\lambda_n(x+a)]}{a+b} \times\left[1 + \sum_{m=1}^{\infty}\left(\prod_{k=1}^{m}\frac{\Gamma[1-\alpha+k(\alpha+\gamma)]}{\Gamma[1+k(\alpha+\gamma)]}\right) \times(-\lambda_n^2 D_{\alpha,\gamma}t^{\alpha+\gamma})^m\right]. \quad (23)$$

We can verify that for $\gamma=0$ we recover the solution of Eq. (1) [7] which is given by

$$W_{\alpha,0}(x,t) = \sum_{n=1}^{\infty}\frac{2\sin[\lambda_n(x_0+a)]\sin[\lambda_n(x+a)]}{a+b}E_{\alpha}[-\lambda_n^2 D_{\alpha,0}t^{\alpha}], \quad (24)$$

where $E_{\alpha}(z)$ is the Mittag-Leffler function defined by $E_{\alpha}(z) = \Sigma_{n=0}^{\infty} z^n/\Gamma[1+n\alpha]$. The MFPT associated to the distribution (24) has been calculated in Ref. [8], in an interval $[0,L]$, by using the following formula [9]:

$$T = \int_{-a}^{b} dx \int_{0}^{\infty} dt W(x,t). \quad (25)$$

As noted by Yuste and Lindenberg [10] the MFPT related to the distribution (24) diverges for $\alpha<1$ and for any length of L.

For $\gamma=\alpha=1/2$ we have the following simplified solution:

$$W_{1/2,1/2}(x,t) = \sum_{n=1}^{\infty}\frac{2\sin[\lambda_n(x_0+a)]\sin[\lambda_n(x+a)]}{a+b} \times\left[1 + \sum_{m=1}^{\infty}\left(\prod_{k=1}^{m}\frac{(2k)!}{(k\,!)^2}\right) \times\frac{\pi^{n/2}}{2^{n(n+1)}}(-\lambda_n^2 D_{1/2,1/2}t)^m\right]. \quad (26)$$

For $\alpha=0$ yields

$$W_{0,\gamma}(x,t) = \sum_{n=1}^{\infty}\frac{2\sin[\lambda_n(x_0+a)]\sin[\lambda_n(x+a)]}{(a+b)[1+\lambda_n^2 D_{0,\gamma}t^{\gamma}]}. \quad (27)$$

The corresponding MFPT is given by

$$T_{0,\gamma} = 4\sum_{n=0}^{\infty}\frac{\sin[\lambda_{2n+1}(x_0+a)]}{(a+b)\lambda_{2n+1}}\int_0^{\infty}\frac{dt}{[1+\lambda_{2n+1}^2 D_{0,\gamma}t^{\gamma}]}. \quad (28)$$

We see that this MFPT diverges for large t, in the interval $\gamma<1$, which corresponds to the subdiffusive regimes. For $\gamma>1$ we obtain

$$T_{0,\gamma} = 4\pi\sum_{n=0}^{\infty}\frac{\sin[\lambda_{2n+1}(x_0+a)]}{\gamma(a+b)\lambda_{2n+1}(\lambda_{2n+1}^2 D_{0,\gamma})^{1/\gamma}\sin\left[\frac{\pi}{\gamma}\right]}. \quad (29)$$

Now we consider $a=0$, $b=L$ and put x_0 in the middle of the interval, i.e., $x_0=L/2$. Then,

$$T_{0,\gamma} = \left[\frac{\frac{1}{4}^2 L^2}{\pi^2 D_{0,\gamma}}\right]^{1/\gamma}\frac{\zeta\left[\frac{2+\gamma}{\gamma},\frac{1}{4}\right]-\zeta\left[\frac{2+\gamma}{\gamma},\frac{3}{4}\right]}{\gamma\sin\left(\frac{\pi}{\gamma}\right)}, \quad (30)$$

where $\zeta(z,q)$ is the generalized Riemann zeta function. In Fig. 1 we plot the MFPT $T_{0,\gamma}$ for different diffusive regimes $(\gamma>1)$. The MFPT shows a monotonic behavior. We should emphasize that our model presents finite results for both the MFPT and mean squared displacement. In contrast with the

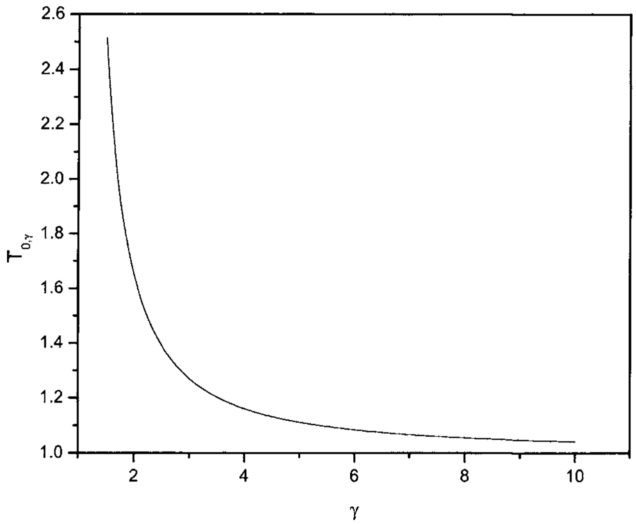

FIG. 1. Plot of the MFPT $T_{0,\gamma}$ in function of the parameter γ for $L^2/(8D_{0,\gamma})=1$.

fractional diffusion equation which cannot lead to the finite results for the MFPT and mean squared displacement, simultaneously [11,12]. Moreover, it should be noted that the solution (30), $T_{0,\gamma}\sim L^{2/\gamma}$, can comply with the relation of the anomalous heat conduction in one-dimensional systems studied in Ref. [11], for superdiffusive regimes. The quantity analyzed in Ref. [11] is the thermal conductivity given by $\kappa=cL^{\beta}$ with the exponent $\beta=2-2/\alpha$, where α is related to the anomalous regime. For our MFPT (30), the parameter α is replaced by γ.

IV. CONCLUSION

In summary, we have investigated the time-fractional diffusion equation with the time dependent diffusion coefficient $D(t)=D_{\alpha,\gamma}t^{\gamma}$. Its second moment is of the power law type with exponent given by $\alpha+\gamma$. They can describe subdiffusive $(0<\alpha+\gamma<1)$, superdiffusive $(\alpha+\gamma>1)$ and localized processes $(\alpha+\gamma<0)$. Further, there are many probability distributions which can describe the normal diffusion with $\alpha+\gamma=1$. We have also obtained the analytical solutions in the infinite and finite domains. We note that the technique used in this work to solve the fractional differential equation is different from other authors. The solutions in the infinite domain, for some specific values of α and γ, have been analyzed. For $\alpha=0$, the probability distribution has the exponential shape. In particular, for $\alpha=0$ and $\gamma=2$ the solution can reproduce the same solution of the usual diffusion equation with the diffusion coefficient dependent on x. In the finite domain we have considered a particle diffusing in a finite interval $[-a,b]$ subject to absorbing boundaries and the initial condition $W(x,0)=\delta(x-x_0)$. For $\alpha=0$ we have obtained the analytical solution for the MFPT. We have shown the MFPT describes a monotonic behavior in function of the parameter γ. An application of the MFPT (30) has been considered. We have shown that the MFPT (30) can describe the exponent of the thermal conductivity studied in Ref. [11], for superdiffusive regimes.

[1] R. Metzer and J. Klafter, Phys. Rep. **339**, 1 (2000); R. Metzer and J. Klafter, J. Phys. A **37**, R161 (2004); A. Carpinteri and F. Mainardi, *Fractals and Fractional Calculus in Continuum Mechanics* (Springer-Verlag, Wien, 1997), pp. 223-276 and pp. 291-348; G. M. Zaslavsky, Phys. Rep. **371**, 461 (2002); F. Mainardi, M. Raberto, R. Gorenflo and E. Scalas, Physica A **287**, 468 (2000); A. J. Majda and P. R. Kramer, Phys. Rep. **314**, 238 (1999).
[2] E. Barkai, R. Metzler, and J. Klafter, Phys. Rev. E **61**, 132 (2000); E. Barkai, Chem. Phys. **284**, 13 (2002).
[3] W. Wyss, J. Math. Phys. **27**, 2782 (1986); W. R. Schneider and W. Wyss, *ibid.* **30**, 134 (1989); W. G. Glöckle and T. F. Nonnenmacher, J. Stat. Phys. **71**, 755 (1993).
[4] G. K. Batchelor, Proc. Cambridge Philos. Soc. **48**, 345 (1952); A. Okubo, J. Oceanogr. Soc. Jpn. **20**, 286 (1962); H. G. E. Hentschel and I. Procaccia, Phys. Rev. A **29**, 1461 (1984).
[5] J. Klafter, A. Blumen, and M. F. Shlesinger, Phys. Rev. A **35**, 3081 (1987).
[6] Kwok Sau Fa and E. K. Lenzi, Phys. Rev. E **67**, 061105 (2003); **71**, 012101 (2005).
[7] G. Rangarajan and M. Ding, Phys. Rev. E **62**, 120 (2000).
[8] M. Gitterman, Phys. Rev. E **62**, 6065 (2000).
[9] C. W. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag, Berlin, 1997).
[10] S. B. Yuste and K. Lindenberg, Phys. Rev. E **69**, 033101 (2004).
[11] B. Li and J Wang, Phys. Rev. Lett. **91**, 044301 (2003); R. Metzler and I. M. Sokolov, *ibid.* **92**, 089401 (2004); B. Li and J. Wang, *ibid.* **92**, 089402 (2004).
[12] I. Goychuk and P. Hänggi, Phys. Rev. E **70**, 051915 (2004).